Eur. Phys. J. C 54, 325–332 (2008)
DOI 10.1140/epjc/s10052-007-0518-x

THE EUROPEAN PHYSICAL JOURNAL C

Regular Article – Theoretical Physics

Path integral representations in noncommutative quantum mechanics and noncommutative version of Berezin–Marinov action

D.M. Gitman[1,a], V.G. Kupriyanov[1,2,b]

[1] Instituto de Física, Universidade de São Paulo, 05315-970 São Paulo, SP, Brazil
[2] Physics Department, Tomsk State University, Russia

Received: 17 October 2007 / Revised version: 3 December 2007 /
Published online: 24 January 2008 –

Abstract. It is known that the actions of field theories on a noncommutative space-time can be written as some modified (we call them θ-modified) classical actions already on the commutative space-time (introducing a star product). Then the quantization of such modified actions reproduces both space-time noncommutativity and the usual quantum mechanical features of the corresponding field theory. In the present article, we discuss the problem of constructing θ-modified actions for relativistic QM. We construct such actions for relativistic spinless and spinning particles. The key idea is to extract θ-modified actions of the relativistic particles from path-integral representations of the corresponding noncommutative field theory propagators. We consider the Klein–Gordon and Dirac equations for the causal propagators in such theories. Then we construct for the propagators path-integral representations. Effective actions in such representations we treat as θ-modified actions of the relativistic particles. To confirm the interpretation, we canonically quantize these actions. Thus, we obtain the Klein–Gordon and Dirac equations in the noncommutative field theories. The θ-modified action of the relativistic spinning particle is just a generalization of the Berezin–Marinov pseudoclassical action for the noncommutative case.

1 Introduction

Recently quantum field theories on a noncommutative space-time have received a lot of attention, see for example [1–4] and references therein. The noncommutative $d+1$ space-time can be realized by the coordinate operators $\hat{q}^\mu$, $\mu = 0, 1, \ldots, d$, satisfying

$$[\hat{q}^\mu, \hat{q}^\nu] = \mathrm{i}\theta^{\mu\nu} , \qquad (1)$$

where, in the general case, the noncommutativity parameters enter in the theory via an antisymmetric matrix $\theta^{\mu\nu}$. Obviously, many of principal problems related to the noncommutativity can be examined already in the noncommutative quantum mechanics (QM). Some of the articles in this direction consider a generalization of the well-known QM problems (harmonic oscillator [5, 6], the Landau problem [7, 8], Lamb shift in the hydrogen atom spectrum [9], a particle in the Aharonov–Bohm field [10, 11], and a system in a central potential [12]) for the noncommutative case, trying to extract possible observable differences with the commutative case. In this connection, the path-integral representations in nonrelativistic QM were studied in [10, 11] for the description of the Aharonov–Bohm effect, and in [13–17] for the calculations of the simple cases of the harmonic oscillator [14] and a free particle [17].

One ought to say that the classical actions of field theories on a noncommutative space-time can be written as some modified classical actions already on the commutative space-time (introducing a star product). Then the quantization of such modified actions (let us call them θ-modified actions in what follows) reproduces both space-time noncommutativity and the usual quantum mechanical features of the corresponding field theory. Considering QM of one particle (or a system of N particles) with noncommutative coordinates, one can ask the question how to construct a θ-modified classical action (with already commuting coordinates) for the system. As in the case of field theory, such θ-modified classical actions in the course of a quantization must reproduce both the noncommutativity of the coordinates and the usual QM features of the corresponding finite-dimensional physical system. For nonrelativistic QM, the latter problem was solved in [21]; see also [22, 23]. In the relativistic case an important role is played by the Poincaré group, whose realization on a noncommutative space-time was recently

[a] e-mail: gitman@dfn.if.usp.br, dmitrygitman@hotmail.com
[b] e-mail: kvg@dfn.if.usp.br

constructed as the twisted Poincaré symmetry in [24, 25]. In the present article we discuss the problem of constructing θ-modified actions for relativistic QM. We construct θ-modified actions for relativistic spinless and spinning particles. The key idea is to extract θ-modified actions of the relativistic particles from path-integral representations of the corresponding noncommutative field theory propagators. We consider θ-modified Klein–Gordon and Dirac equations with external backgrounds for the causal propagators. Then, using techniques developed in [18, 19] for the usual commutative case, we construct for them path-integral representations. The effective actions in such path-integral representations we treat as θ-modified actions of the relativistic particles. To confirm this interpretation, we canonically quantize these actions. Thus, we obtain the above mentioned θ-modified Klein–Gordon and Dirac equations. The θ-modified action of the relativistic spinning particle is a generalization of the Berezin–Marinov pseudoclassical action [20] for the noncommutative case. One ought to say that the effects of the noncommutativity appear to be essential only due to the external background. Finally, we consider a noncommutative d-dimensional nonrelativistic QM with no restrictions on the noncommutativity parameters $\theta^{\mu\nu}$ and a formally arbitrary Hamiltonian. We construct a path-integral representation for the corresponding propagation function and demonstrate that the effective action in our path-integral representation is just the θ-modified action for nonrelativistic QM proposed in [21–23].

2 Path integral representations for particle propagators in noncommutative field theory

2.1 Spinless case

In field theories the effect of the noncommutativity of the space-time can be realized by a substitution of the usual function product by the Weyl–Moyal star product

$$f(x)*g(x)=f(x)\exp\left\{\frac{\mathrm{i}}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu\right\}g(x)\,, \qquad (2)$$

where $f(x)$ and $g(x)$ are two arbitrary infinitely differentiable functions of the commutative variables x^μ.

The action of a noncommutative field theory of a scalar field Φ that interacts with an external electromagnetic field $A_\mu(x)$ reads

$$S^\theta_{\text{scal-field}}=\int \mathrm{d}^Dx\left[(P_\mu*\Phi)*(P^\mu*\bar{\Phi})+m^2\Phi\bar{\Phi}\right],$$
$$P_\mu=\mathrm{i}\partial_\mu-gA_\mu(x)\,. \qquad (3)$$

The corresponding Euler–Lagrange equation,

$$\frac{\delta S^\theta_{\text{scal-field}}}{\delta\bar{\Phi}}=0\Longrightarrow\left[P_\mu*P^\mu-m^2\right]*\Phi=0\,, \qquad (4)$$

being rewritten with the help of (2) takes the form

$$(\tilde{P}^2-m^2)\Phi=0\,,\quad \tilde{P}^2=\tilde{P}_\mu\tilde{P}^\mu\,, \qquad (5)$$
$$\tilde{P}_\mu=\mathrm{i}\partial_\mu-gA_\mu\left(x^\mu+\frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\nu\right), \qquad (6)$$

and it is an analog of the Klein–Gordon equation for the noncommutative case. It is supposed that the operator function of two noncommuting operators in (6) is Weyl ordered. The propagator in the noncommutative scalar field theory is the causal Green function $D^{\mathrm{c}}(x,y)$ of (5),

$$(\tilde{P}^2-m^2)D^{\mathrm{c}}(x,y)=-\delta(x-y)\,. \qquad (7)$$

From this point on, we are going to follow the way elaborated in [18] to construct a path-integral representation for the propagator: we consider $D^{\mathrm{c}}(x,y)$ as a matrix element of an operator $\hat{D}^{\mathrm{c}}$ in a Hilbert space:

$$D^{\mathrm{c}}(x,y)=\langle x|\hat{D}^{\mathrm{c}}|y\rangle\,. \qquad (8)$$

Here $|x\rangle$ are the eigenvectors of some self-adjoint and mutually commuting operators $\hat{x}^\mu$,

$$\hat{x}^\mu=\hat{q}^\mu+\frac{1}{2\hbar}\theta^{\mu\nu}\hat{p}_\nu\,, \qquad (9)$$

where the operators $\hat{q}^\mu$ obey the commutation relations (1), and $\hat{p}_\mu$ are the momentum operators conjugate to $\hat{x}^\mu$,

$$\left[\hat{x}^\mu,\hat{p}_\nu\right]=\mathrm{i}\hbar\delta^\mu_\nu\,,\quad [\hat{x}^\mu,\hat{x}^\nu]=[\hat{p}_\mu,\hat{p}_\nu]=0\,,$$
$$\hat{x}^\mu|x\rangle=x^\mu|x\rangle\,,\qquad \langle x|y\rangle=\delta^D(x-y)\,,\qquad \int|x\rangle\langle x|\,\mathrm{d}x=I\,; \qquad (10)$$

the change of variables (9) was first used in the context of noncommutative QM in [9]. Then (7) implies $\hat{D}^{\mathrm{c}}=(m^2-\Pi^2)^{-1}$, where[1]

$$\hat{\Pi}_\mu=-\hat{p}_\mu-gA_\mu(\hat{q})\,,\quad [\hat{\Pi}_\mu,\hat{\Pi}_\nu]=-\mathrm{i}g\hat{F}_{\mu\nu}\,,$$
$$\hat{F}_{\mu\nu}=\partial_\mu A_\nu(\hat{q})-\partial_\nu A_\mu(\hat{q})+\mathrm{i}g[A_\mu(\hat{q}),A_\nu(\hat{q})]\,. \qquad (11)$$

Due to the star product property $f(\hat{q})g(\hat{q})=(f*g)(\hat{q})$, we can represent the operator $\hat{F}_{\mu\nu}$ as follows:

$$\hat{F}_{\mu\nu}=F^*_{\mu\nu}(\hat{q})\,,$$
$$F^*_{\mu\nu}(q)=\partial_\mu A_\nu-\partial_\nu A_\mu+\mathrm{i}g(A_\mu*A_\nu-A_\nu*A_\mu)\,. \qquad (12)$$

Using the Schwinger proper-time representation for the inverse operator, we get

$$D^{\mathrm{c}}=D^{\mathrm{c}}(x_{\text{out}},x_{\text{in}})=\mathrm{i}\int_0^\infty\langle x_{\text{out}}|\exp\left[-\frac{\mathrm{i}}{\hbar}\hat{\mathcal{H}}(\lambda)\right]|x_{\text{in}}\rangle\,\mathrm{d}\lambda\,, \qquad (13)$$
$$\hat{\mathcal{H}}(\lambda)=\lambda(m^2-\Pi^2)\,.$$

[1] Here and in what follows $\Pi^2=\Pi_\mu\Pi^\mu$ and so on.

Here and in what follows the infinitesimal factor $-\mathrm{i}\epsilon$ is included in m^2. Doing finally a discretization, similar to that in [18], we get a path-integral representation for the propagator (13):

$$D^{\mathrm{c}} = \mathrm{i}\int_0^\infty \mathrm{d}\lambda_0 \int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}} \mathrm{D}x \int_{\lambda_0} \mathrm{D}\lambda \times \int \mathrm{D}p\,\mathrm{D}\pi \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S^\theta_{\text{scal-part}} + S_{\mathrm{GF}}\right]\right\}, \quad (14)$$

where

$$\begin{aligned} S^\theta_{\text{scal-part}} &= \int_0^1 \left[\lambda(\mathcal{P}^2 - m^2) + p_\mu \dot{x}^\mu\right]\mathrm{d}\tau\,, \\ S_{\mathrm{GF}} &= \int_0^1 \pi\dot{\lambda}\,\mathrm{d}\tau\,, \\ \mathcal{P}_\mu &= -p_\mu - gA_\mu\left(x^\mu - \frac{1}{2\hbar}\theta^{\mu\nu}p_\nu\right), \\ \dot{x} &= \frac{\mathrm{d}x}{\mathrm{d}\tau}\,, \quad \dot{\lambda} = \frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\,. \end{aligned} \quad (15)$$

The functional integration in (14) goes over trajectories $x^\mu(\tau)$, $p_\mu(\tau)$, $\lambda(\tau)$, and $\pi(\tau)$, parametrized by some invariant parameter $\tau \in [0,1]$ and obeying the boundary conditions $x(0) = x_{\mathrm{in}}$, $x(1) = x_{\mathrm{out}}$, $\lambda(0) = \lambda_0$.

Since the momenta are involved in the arguments of the electromagnetic potentials A_μ, an integration over the momenta in the representation (14) is difficult to perform in the general case. On the other hand, we can go over from x to new coordinates q,

$$q^\mu = x^\mu - \frac{1}{2\hbar}\theta^{\mu\nu}p_\nu\,, \quad (16)$$

which correspond in a sense to the noncommutative operators $\hat{q}^\mu$ (1). Then

$$\begin{aligned} D^{\mathrm{c}} &= \mathrm{i}\int_0^\infty \mathrm{d}\lambda_0 \int_{x_{\mathrm{in}}-\theta p/2\hbar}^{x_{\mathrm{out}}-\theta p/2\hbar} \mathrm{D}q \int_{\lambda_0} \mathrm{D}\lambda \\ &\quad \times \int \mathrm{D}p\,\mathrm{D}\pi \exp\left\{\frac{\mathrm{i}}{\hbar}S^\theta_{\text{scal-part}} + S_{\mathrm{GF}}\right\}, \\ S^\theta_{\text{scal-part}} &= \int_0^1 \left\{\lambda\left[(p_\mu + gA_\mu(q))^2 - m^2\right] + p_\mu \dot{q}^\mu + \frac{1}{2\hbar}\dot{p}_\mu\theta^{\mu\nu}p_\nu\right\}\mathrm{d}\tau\,. \end{aligned} \quad (17)$$

Thus, we get rid from the above mentioned difficulty but a new one has appeared. The action $S^\theta_{\text{scal-part}}$ in (17) contains an "inconvenient" term $\dot{p}_\mu\theta^{\mu\nu}p_\nu/2\hbar$. Here the possibility to integrate over the momenta is related to the study of the structure of $\theta^{\mu\nu}$ matrix and with a subsequent transition to some Darboux coordinates.

The representation (17) can be treated as a Hamiltonian path integral for the scalar particle propagator in the noncommutative field theory. The exponent in the integrand (17) can be considered as an effective and nondegenerate Hamiltonian action of a scalar particle in a noncommutative space-time. It consists of two parts. The first one S_{GF} can be treated as a gauge fixing term and corresponds, in fact, to the gauge condition $\dot{\lambda} = 0$. The rest part of the effective action $S^\theta_{\text{scal-part}}$ can be treated as θ-modification of the usual Hamiltonian action of a spinless relativistic particle in the commutative case. This action differs from the corresponding commutative case [18] by the term $\frac{1}{2\hbar}\dot{p}_\mu\theta^{\mu\nu}p_\nu$.

2.2 Spinning particle

Consider a θ-modified action of noncommutative field theory of a spinor field Ψ that interacts with an external electromagnetic background A_μ. Being written in commuting D-dimensional Minkowski coordinates x^μ, $\mu = 0,1,\ldots,D-1$, the action reads

$$S^\theta_{\text{spinor-field}} = \int \mathrm{d}x^D \bar{\Psi} * \left(P_\mu\gamma^\mu + m\right) * \Psi\,, \quad (18)$$

where γ^μ are gamma matrices in D dimensions, $[\gamma^\mu,\gamma^\nu]_+ = 2\eta^{\mu\nu}$. In this article, we consider D to be even, $D = 2d$, for simplicity, and we consider it a generalization of 4-dimensional Minkowski space; the odd case can be considered in the same manner following the ideas of [19]. As is known [28], in even dimensions a matrix representation of the Clifford algebra with dimensionality $\dim\gamma^\mu = 2^d$ always exists. In other words, γ^μ are $2^d \times 2^d$ matrices. In such dimensions one can introduce another matrix, $\gamma^{D+1} = r\gamma^0\gamma^1\ldots\gamma^{D-1}$, where $r = 1$, if d is even, and $r = \mathrm{i}$, if d is odd, which anticommutes with all γ^μ (analog of γ^5 in four dimensions), $[\gamma^{D+1},\gamma^\mu]_+ = 0$ and $(\gamma^{D+1})^2 = -1$. The Euler–Lagrange equations

$$\frac{\delta S^\theta_{\text{spinor-field}}}{\delta\bar{\Psi}} = 0 \longrightarrow \left(P_\mu\gamma^\mu + m\right) * \Psi = 0\,, \quad (19)$$

being rewritten with the help of (2), take the form

$$(\tilde{P}_\mu\gamma^\mu - m)\Psi = 0\,, \quad \tilde{P}_\mu = \mathrm{i}\partial_\mu - gA_\mu\left(x^\mu + \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\nu\right) \quad (20)$$

and represent an analog of the Dirac equation for the noncommutative case. The propagator of the noncommutative spinor field theory is the causal Green function $G^{\mathrm{c}}(x,y)$ of (20),

$$\left(\tilde{P}_\mu\gamma^\mu - m\right)G^{\mathrm{c}}(x,y) = -\delta^D(x-y)\,. \quad (21)$$

Following [18, 19], we pass to a θ-modified Dirac operator that is homogeneous in the γ matrices. Indeed, let us rewrite (21) in terms of the propagator $\tilde{G}^{\mathrm{c}}(x,y)$ transformed by γ^{D+1},

$$\begin{aligned} \tilde{G}^{\mathrm{c}}(x,y) &= G^{\mathrm{c}}(x,y)\gamma^{D+1}\,, \\ \left(\tilde{P}_\mu\tilde{\gamma}^\mu - m\gamma^{D+1}\right)\tilde{G}^{\mathrm{c}}(x,y) &= \delta^D(x-y)\,, \end{aligned} \quad (22)$$

where $\tilde{\gamma}^\mu = \gamma^{D+1}\gamma^\mu$. The matrices $\tilde{\gamma}^\mu$ have the same commutation relations as the initial ones without tilde, $[\tilde{\gamma}^\mu,\tilde{\gamma}^\nu]_+ = 2\eta^{\mu\nu}$, and they anticommute with the matrix

γ^{D+1}. The set of $D+1$ gamma matrices $\tilde{\gamma}^{\nu}$ and γ^{D+1} form a representation of the Clifford algebra in an odd number, $2d+1$, of dimensions. Let us denote such matrices via Γ^n,

$$\Gamma^n = \begin{cases} \tilde{\gamma}^{\mu}, & n=\mu=0,\dots,D-1, \\ \gamma^{D+1}, & n=D \end{cases} \tag{23}$$
$$[\Gamma^k, \Gamma^n]_+ = 2\eta^{kn},$$
$$\eta_{kn} = \mathrm{diag}(\underbrace{1,-1,\dots,-1}_{D+1}), \quad k,n=0,\dots,D.$$

In terms of these matrices (22) takes the form

$$\tilde{P}_n \Gamma^n \tilde{G}^{\mathrm{c}}(x,y) = \delta^D(x-y),$$
$$\tilde{P}_{\mu} = \mathrm{i}\partial_{\mu} - gA_{\mu}\left(x^{\mu} + \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_{\nu}\right), \quad \tilde{P}_D = -m. \tag{24}$$

Now again, similar to (8), we present $\tilde{G}^{\mathrm{c}}(x,y)$ as a matrix element of an operator $\hat{G}^{\mathrm{c}}$ (in the coordinate representation (10)),

$$\tilde{G}^{\mathrm{c}}_{ab}(x,y) = \langle x|\hat{G}^{\mathrm{c}}_{ab}|y\rangle, \quad a,b=1,2,\dots,2^d, \tag{25}$$

where the spinor indices a,b are written here explicitly for clarity and will be omitted hereafter. Equation (24) implies $\hat{S}^{\mathrm{c}} = (\Pi_n\Gamma^n)^{-1}$, where Π_{μ} are defined in (11), and $\Pi_D = -m$. Using a generalization of the Schwinger proper-time representation proposed in [18], we write the Green function (25) in the form

$$\tilde{G}^{\mathrm{c}} = \tilde{G}^{\mathrm{c}}(x_{\mathrm{out}}, x_{\mathrm{in}}) = \int_0^{\infty} \mathrm{d}\lambda \int \langle x_{\mathrm{out}}|\mathrm{e}^{-\mathrm{i}\hat{\mathcal{H}}(\lambda,\chi)}|x_{\mathrm{in}}\rangle\,\mathrm{d}\chi, \tag{26}$$
$$\hat{\mathcal{H}}(\lambda,\chi) = \lambda\left(m^2 - \Pi^2 + \frac{\mathrm{i}g}{2}F_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu}\right) + \Pi_n\Gamma^n\chi.$$

Similar to [18], we present the matrix element entering in the expression (26) by means of a Hamiltonian path integral

$$\tilde{G}^{\mathrm{c}} = \exp\left(\mathrm{i}\Gamma^n\frac{\partial_l}{\partial\varepsilon^n}\right)\int_0^{\infty}\mathrm{d}\lambda_0\int\mathrm{d}\chi_0\int_{\lambda_0}\mathrm{D}\lambda\int_{\chi_0}\mathrm{D}\chi$$
$$\times\int_{x_{\mathrm{in}}}^{x_{\mathrm{out}}}\mathrm{D}x\int\mathrm{D}p\int\mathrm{D}\pi\int\mathrm{D}\nu\int_{\psi(0)+\psi(1)=\varepsilon}\mathcal{D}\psi$$
$$\times\exp\left\{\mathrm{i}\int_0^1\left[\lambda\left(\mathcal{P}^2 - m^2 + 2\mathrm{i}gF^*_{\mu\nu}\psi^{\mu}\psi^{\nu}\right) + 2\mathrm{i}\mathcal{P}_n\psi^n\chi\right.\right.$$
$$\left.\left.-\mathrm{i}\psi_n\dot{\psi}^n + p_{\mu}\dot{x}^{\mu} + \pi\dot{\lambda} + \nu\dot{\chi}\right]\mathrm{d}\tau + \psi_n(1)\psi^n(0)\right\}\Big|_{\varepsilon=0}. \tag{27}$$

Here ε^n are odd variables, anticommuting with the Γ-matrices, and $\partial_l/\partial\varepsilon^n$ denotes the left Grassmann derivative,

$$\mathcal{P}_{\mu} = -p_{\mu} - gA_{\mu}\left(x^{\mu} - \frac{1}{2\hbar}\theta^{\mu\nu}p_{\nu}\right), \quad \mathcal{P}_D = -m,$$
$$F^*_{\mu\nu} = F^*_{\mu\nu}\left(x^{\mu} - \frac{1}{2\hbar}\theta^{\mu\nu}p_{\nu}\right),$$

the function $F^*_{\mu\nu}(q)$ is defined in (12), and the integration goes over even trajectories $x(\tau)$, $p(\tau)$, $\lambda(\tau)$, $\pi(\tau)$, and odd trajectories $\psi_n(\tau)$, $\chi(\tau)$, $\nu(\tau)$, parametrized by some invariant parameter $\tau\in[0,1]$ and obeying the boundary conditions $x(0)=x_{\mathrm{in}}$, $x(1)=x_{\mathrm{out}}$, $\lambda(0)=\lambda_0$, $\chi(0)=\chi_0$.

Performing the change of variables (16) in (27), we obtain another representation for $\tilde{G}^{\mathrm{c}}$,

$$\tilde{G}^{\mathrm{c}} = \exp\left(\mathrm{i}\Gamma^n\frac{\partial_l}{\partial\varepsilon^n}\right)\int_0^{\infty}\mathrm{d}\lambda_0\int\mathrm{d}\chi_0\int_{\lambda_0}\mathrm{D}\lambda\int_{\chi_0}\mathrm{D}\chi$$
$$\times\int_{-\infty}^{\infty}\mathrm{D}p\int_{x_{\mathrm{in}}-\theta p/2\hbar}^{x_{\mathrm{out}}-\theta p/2\hbar}\mathrm{D}q\int\mathrm{D}\pi\int\mathrm{D}\nu\int_{\psi(0)+\psi(1)=\varepsilon}\mathcal{D}\psi$$
$$\times\exp\left\{\mathrm{i}\left[S^{\theta}_{\text{spin-part}} + S_{\mathrm{GF}}\right] + \psi_n(1)\psi^n(0)\right\}\Big|_{\varepsilon=0}, \tag{28}$$

where

$$S^{\theta}_{\text{spin-part}} = \int_0^1\left[\lambda\left((p_{\mu}+gA_{\mu})^2 - m^2 + 2\mathrm{i}gF^*_{\mu\nu}\psi^{\mu}\psi^{\nu}\right)\right.$$
$$+ 2\mathrm{i}(p_{\mu}+gA_{\mu}(q))\psi^{\mu}\chi - 2\mathrm{i}m\psi^D\chi$$
$$\left.-\mathrm{i}\psi_n\dot{\psi}^n + p_{\mu}\dot{q}^{\mu} + \frac{1}{2\hbar}\dot{p}_{\mu}\theta^{\mu\nu}p_{\nu}\right]\mathrm{d}\tau, \tag{29a}$$
$$S_{\mathrm{GF}} = \int_0^1(\pi\dot{\lambda} + \nu\dot{\chi})\,\mathrm{d}\tau. \tag{29b}$$

Note that in [29] an attempt was made to construct the path integral representation of the Green function of the noncommutative Dirac equation. However, the consideration was perturbative in θ (taking into account only the first-order perturbation). As a consequence the authors did not obtain the corresponding action (29a); moreover, the essential term $\dot{p}_{\mu}\theta^{\mu\nu}p_{\nu}/2\hbar$ was missing.

3 Pseudoclassical action of spinning particle in noncommutative space-time

Similar to the spinless case, the exponent in the integrand (28) can be considered as an effective and nondegenerate Hamiltonian action of a spinning particle in noncommutative space-time. It consists of two principal parts. The first one S_{GF} with derivatives of λ and χ can be treated as a gauge fixing term, which corresponds to the gauge conditions $\dot{\lambda}=\dot{\chi}=0$. The rest part $S^{\theta}_{\text{spin-part}}$ can be treated as a gauge invariant action of a spinning particle in noncommutative space-time. The action $S^{\theta}_{\text{spin-part}}$ is a θ-modification of the Hamiltonian form of the Berezin–Marinov action [20]. It will be studied and quantized below to justify such an interpretation.

One can easily verify that $S^{\theta}_{\text{spin-part}}$ is reparametrization invariant. The explicit form of supersymmetry transformations, which generalize ones for the Berezin–Marinov action, is not so easily to derive. Their presence will be proved in an indirect way. Namely, we are going to prove the existence of two primary first-class constraints in the corresponding Hamiltonian formulation.

Let us consider $S^{\theta}_{\text{spin-part}}$ as a Lagrangian action with generalized coordinates $Q_A = (q^{\mu}, p_{\mu})$, $A=(\zeta,\mu)$, $\zeta=1,2$,

$Q_{1\mu} = q^\mu$, $Q_{2\mu} = p_\mu$; χ, ψ, and λ, and let us perform a Hamiltonization of such an action. To this end, we introduce the canonical momenta P conjugate to the generalized coordinates as follows:

$$P_{Q_A} = \frac{\partial L}{\partial \dot{Q}^A} = J_A(q)\,, \quad J_{1\mu} = p_\mu\,, \quad J_{2\mu} = \frac{1}{2\hbar}\theta^{\mu\nu} p_\nu\,,$$
$$P_\lambda = \frac{\partial L}{\partial \dot{\lambda}} = 0\,, \quad P_\chi = \frac{\partial_r L}{\partial \dot{\chi}} = 0\,, \quad P_n = \frac{\partial_r L}{\partial \dot{\psi}^n} = -\mathrm{i}\psi_n\,. \tag{30}$$

It follows from (30) that there exist primary constraints $\Phi^{(1)} = 0$,

$$\Phi_l^{(1)} = \begin{cases} \Phi_{1A}^{(1)} = P_A - J_A(q)\,, \\ \Phi_2^{(1)} = P_\lambda\,, \quad \Phi_3^{(1)} = P_\chi\,, \\ \Phi_{4n}^{(1)} = P_n + \mathrm{i}\psi_n\,. \end{cases} \tag{31}$$

The Poisson brackets of the primary constraints are

$$\{\Phi_{1A}^{(1)}, \Phi_{1B}^{(1)}\} = \Omega_{AB} = \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbb{I} & \theta/\hbar \end{pmatrix}, \quad \{\Phi_{4n}^{(1)}, \Phi_{4m}^{(1)}\} = 2\mathrm{i}\eta_{nm}\,,$$
$$\{\Phi_{1A}^{(1)}, \Phi_{4n}^{(1)}\} = \{\Phi_{1A}^{(1)}, \Phi_{2,3}^{(1)}\} = \{\Phi_{4n}^{(1)}, \Phi_{2,3}^{(1)}\} = 0\,, \tag{32}$$

where $\theta = \theta^{\mu\nu}$, $\mathbb{I}$ is a $D \times D$ unit matrix, and $\mathbf{0}$ denotes a $D \times D$ zero matrix. Note that $\det \Omega_{AB} = 1$, and

$$\omega^{AB} = \Omega_{AB}^{-1} = \begin{pmatrix} \theta/\hbar & -\mathbb{I} \\ \mathbb{I} & \mathbf{0} \end{pmatrix}.$$

Now we construct the total Hamiltonian $H^{(1)}$, according to the standard procedure [33]. Thus, we obtain

$$H^{(1)} = H + \Lambda_l \Phi_l^{(1)}\,,$$
$$H = -\lambda\left[(p_\mu + gA_\mu)^2 - m^2 + 2\mathrm{i}gF_{\mu\nu}^*(q)\psi^\mu\psi^\nu\right] + 2\mathrm{i}\chi\left((p_\mu + gA_\mu)\psi^\mu - m\psi^D\right), \tag{33}$$

where Λ_l are the corresponding Lagrangian multipliers. The consistency conditions $\dot{\Phi}_{1A,4n}^{(1)} = \{\Phi_{1A,4n}^{(1)}, H^{(1)}\} = 0$ for the primary constraints $\Phi_{1A}^{(1)}$ and $\Phi_{4n}^{(1)}$ allow us to fix the Lagrange multipliers λ^{1A} and λ^{4n}. The consistency conditions for the constraints $\Phi_{2,3}^{(1)}$ imply secondary constraints $\Phi_{1,2}^{(2)} = 0$,

$$\Phi_1^{(2)} = (p_\mu + gA_\mu)\psi^\mu - m\psi^D = 0\,, \tag{34}$$
$$\Phi_2^{(2)} = (p_\mu + gA_\mu)^2 - m^2 + 2\mathrm{i}gF_{\mu\nu}^*\psi^\mu\psi^\nu = 0\,, \tag{35}$$

and there are no other constraints. Thus, the Hamiltonian H appears to be proportional to constraints, as always in the case of a reparametrization invariant theory,

$$H = 2\mathrm{i}\chi\Phi_1^{(2)} - \lambda\Phi_2^{(2)}\,.$$

No more secondary constraints arise from the Dirac procedure, and the Lagrange multipliers λ^2 and λ^3 remain undetermined, in perfect correspondence with the fact that the number of gauge transformations parameters equals two for the theory in question.

One can go over from the initial set of constraints $(\Phi^{(1)}, \Phi^{(2)})$ to the equivalent one $(\Phi^{(1)}, T)$, where:

$$T = \Phi^{(2)} + \frac{\partial \Phi^{(2)}}{\partial q^A}\omega^{AB}\Phi_{1B}^{(1)} + \frac{\mathrm{i}}{2}\frac{\partial_r \Phi^{(2)}}{\partial \psi^n}\Phi_{4n}^{(1)}\,. \tag{36}$$

The new set of constraints can be explicitly divided in a set of first-class constraints, which is $(\Phi_{2,3}^{(1)}, T)$ and in a set of second-class constraints, which is $(\Phi_{1A}^{(1)}, \Phi_{4n}^{(1)})$.

Now we consider an operator quantization. To this end we perform a partial gauge fixing, imposing gauge conditions $\Phi_{1,2}^{\mathrm{G}} = 0$ to the primary first-class constraints $\Phi_{1,2}^{(1)}$,

$$\Phi_1^{\mathrm{G}} = \chi = 0\,, \quad \Phi_2^{\mathrm{G}} = \lambda = 1/m\,. \tag{37}$$

One can check that the consistency conditions for the gauge conditions (37) lead to fixing the Lagrange multipliers λ_2 and λ_3. Thus, at this stage we reduced our Hamiltonian theory to one with the first-class constraints T and second-class ones $\varphi = (\Phi^{(1)}, \Phi^{\mathrm{G}})$. Then, we apply the so called Dirac method for systems with first-class constraints [34], which, being generalized to the presence of second-class constraints, can be formulated as follows: the commutation relations between operators are calculated according to the Dirac brackets with respect to the second-class constraints only; second-class constraints as operators equal zero; first-class constraints as operators are not zero but are considered in the sense of restrictions on state vectors. All the operator equations have to be realized in a Hilbert space.

The subset of the second-class constraints $(\Phi_{2,3}^{(1)}, \Phi^{\mathrm{G}})$ has a special form [33],[2] so that one can use it for eliminating the variables λ, P_λ, χ and P_χ from the consideration; and then, for the rest of the variables q, p and ψ^n the Dirac brackets with respect to the constraints φ reduce to ones with respect to the constraints $\Phi_{1A}^{(1)}$ and $\Phi_{4n}^{(1)}$ only and can easily be calculated,

$$\{Q^A, Q^B\}_{D(\Phi^{(1)})} = \omega^{AB}\,, \quad \{\psi^n, \psi^m\}_{D(\Phi^{(1)})} = \frac{\mathrm{i}}{2}\eta^{nm}\,,$$

while all other Dirac brackets vanish. Thus, the commutation relations for the operators $\hat{q}^\mu, \hat{p}_\mu, \hat{\psi}^n$, which correspond to the variables q^μ, p_μ, ψ^n respectively, are

$$[\hat{q}^\mu, \hat{p}_\nu]_- = \mathrm{i}\hbar\omega^{\mu, D+\nu} = \mathrm{i}\hbar\delta_\nu^\mu\,,$$
$$[\hat{q}^\mu, \hat{q}^\nu] = \mathrm{i}\hbar\omega^{\mu\nu} = \mathrm{i}\theta^{\mu\nu}\,, \quad [\hat{p}_\mu, \hat{p}_\nu] = 0\,,$$
$$[\hat{\psi}^m, \hat{\psi}^n]_+ = \mathrm{i}\{\psi^m, \psi^n\}_{D(\Phi^{(1)})} = -\frac{1}{2}\eta^{mn}\,. \tag{38}$$

Besides, the following operator equations hold:

$$\hat{\Phi}_{1A}^{(1)} = \hat{P}_A - J_A(\hat{Q})\,, \quad \hat{\Phi}_{4n}^{(1)} = \hat{P}_n + \mathrm{i}\hat{\psi}_n = 0\,. \tag{39}$$

[2] If a part of all the second-class constraints of a theory is a set of second-class constraints of a special form, we can use the latter to reduce the phase-space such that the initial Dirac brackets are reduced to ones in the reduced phase-space with respect to the rest of second-class constraints.

Taking that into account, one can construct a realization of the commutation relations (38) in a Hilbert space whose elements Ψ are 2^d-component columns dependent only on x, such that

$$\hat{q}^\mu = \left(x^\mu + \frac{\mathrm{i}}{2}\theta^{\mu\nu}\partial_\nu\right)\mathbf{I}\,, \quad \hat{p}_\mu = -\mathrm{i}\partial_\mu \mathbf{I}\,, \quad \hat{\psi}^n = \frac{\mathrm{i}}{2}\Gamma^n\,, \tag{40}$$

where $\mathbf{I}$ is the $2^d \times 2^d$ unit matrix, and Γ^n are gamma matrices; see (23). The first-class constraints $\hat{T}$ as operators have to annihilate physical vectors; by virtue of (39) and (36) that implies the equations

$$\hat{\Phi}_1^{(2)}\Psi = 0\,, \quad \hat{\Phi}_2^{(2)}\Psi = 0\,, \tag{41}$$

where $\hat{\Phi}_{1,2}^{(2)}$ are operators that correspond to the constraints (34) and (35). Taking into account the realizations of the commutation relations (38), one easily can see that the first equation of (41) takes the form of the θ-modified Dirac equation,

$$\left(\tilde{P}_\mu\tilde{\gamma}^\mu - m\gamma^{D+1}\right)\Psi = 0 \Longleftrightarrow \left(P_\mu\gamma^\mu + m\right) * \Psi = 0\,, \tag{42}$$

Since $\hat{\Phi}_2^{(2)} = (\hat{\Phi}_1^{(2)})^2$, the second equation of (41) is a consequence of the first one.

Thus, we have constructed a θ-modification of the Berezin–Marinov action (29a) that, being quantized, leads to a quantum theory based on the θ-modified Dirac equation.

Note that space-time non-commutativity $[\hat{q}^0, \hat{q}^i] = \mathrm{i}\theta^{0i}$ can be obtained also from the canonical quantization of the conventional Lagrangian action of a relativistic spinless particle by imposing the special gauge condition $\Phi_{gf} = x^0 + \theta^{0i}p_i - \tau = 0$ [35].

4 Path integral in nonrelativistic quantum mechanics on a noncommutative space

In this section, we construct a path-integral representation for the propagation function (a symbol of the evolution operator) in nonrelativistic QM on a noncommutative space. We compare our result with some previous constructions and use it to extract a θ-modified first-order classical Hamiltonian action for such a system.

We consider a d-dimensional nonrelativistic QM with basic canonical operators of coordinates $\hat{q}^k$ and momenta $\hat{p}_j$, $k, j = 1, \ldots, d$ that obey the following commutation relations:

$$[\hat{q}^k, \hat{q}^j] = \mathrm{i}\theta^{kj}\,, \quad \left[\hat{q}^k, \hat{p}_j\right] = \mathrm{i}\hbar\delta_j^k\,, \quad [\hat{p}_k, \hat{p}_j] = 0\,. \tag{43}$$

It is supposed that the nonvanishing commutation relations for the coordinate operators in (43) have emerged from the noncommutative properties of the position space. The time evolution of the system under consideration is governed by a self-adjoint Hamiltonian $\hat{H}$. We believe that behind such a QM there exists a classical theory with a θ-modified action (which we are going to restore in what follows), such that the quantization of this action leads to the QM.

In conventional nonrelativistic QM, one constructs a path-integral representation for matrix elements (in a coordinate representation) of the evolution operator $\hat{U}(t,t')$. In the QM under consideration, we also start with such an operator. It obeys the Schrödinger equation and for time independent $\hat{H}$ (which we consider for simplicity in what follows) has the form

$$\hat{U}(t', t) = \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}(t'-t)\right\}. \tag{44}$$

Since the coordinate operators $\hat{q}$ do not commute, they do not possess a common complete set of eigenvectors. Therefore, there is no q-coordinate representation and one cannot speak of matrix elements of the evolution operator in such a representation. Consequently, one cannot define a probability amplitude of a transition between two points in the position space. Nevertheless, one can consider other types of matrix elements of the evolution operator that are probability amplitudes (evolution functions) and that can be represented via path integrals. Below, we consider two types of such matrix elements,

$$G_p = \langle p^{\text{out}}|\hat{U}(t_{\text{out}}, t_{\text{in}})|p^{\text{in}}\rangle\,, \\ G_x = \langle x_{\text{out}}|\hat{U}(t_{\text{out}}, t_{\text{in}})|x_{\text{in}}\rangle\,. \tag{45}$$

In (45) $|p\rangle$ is a complete set of eigenvectors of commuting operators $\hat{p}$,

$$\hat{p}_j|p\rangle = p_j|p\rangle\,, \quad \langle p|p'\rangle = \delta(p-p')\,, \\ \int |p\rangle\langle p|\,\mathrm{d}p = I\,, \quad \mathrm{d}p = \prod_i \mathrm{d}p_i\,, \\ \langle p|x\rangle = \frac{1}{(2\pi\hbar)^{d/2}}\exp\left\{-\frac{\mathrm{i}}{\hbar}p_i x^i\right\}, \\ \langle p|\hat{x}|p'\rangle = \mathrm{i}\hbar\frac{\partial}{\partial p}\langle p|p'\rangle\,, \tag{46}$$

and $|x\rangle$ is a complete set of eigenvectors of some commuting operators $\hat{x}^k$ canonically conjugate to $\hat{p}$. We chose these operators as follows[3]:

$$\hat{x}^k = \hat{q}^k + \frac{\theta^{kj}\hat{p}_j}{2\hbar}\,, \quad [\hat{x}^k, \hat{x}^j] = 0\,, \quad [\hat{x}^k, \hat{p}_j] = \mathrm{i}\hbar\delta_j^k\,, \\ \hat{x}^\mu|x\rangle = x^\mu|x\rangle\,, \quad \langle x|y\rangle = \delta^D(x-y)\,, \\ \int |x\rangle\langle x|\,\mathrm{d}x = I\,, \quad \mathrm{d}x = \prod_i \mathrm{d}x^i\,. \tag{47}$$

First, let us construct a path-integral representation for the evolution function G_p. To this end, as usual, we divide the time interval $T = t_{\text{out}} - t_{\text{in}}$ in N equal parts $\Delta t = T/N$ by means of the points t_k, $k = 1, \ldots, N-1$, such

[3] For the first time the commuting operators $\hat{x}^k$ were introduced in [9].

that $t_k = t_{\rm in} + k\Delta t$. Using the group property of the evolution operator and the completeness relation (see (46)) for the set $|p\rangle$, one can write

$$G_p = \lim_{N\to\infty} \int_{-\infty}^{\infty} \mathrm{d}p^{(1)} \dots \mathrm{d}p^{(N-1)} \times \prod_{k=1}^{N} \langle p^{(k)}| \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}(t_k - t_{k-1})\right\}|p^{(k-1)}\rangle\,, \tag{48}$$

where $p^{(0)} = p^{\rm in}$, $p^{(N)} = p^{\rm out}$, and $p^{(k)} = (p_i^{(k)})$. Bearing in mind the limiting process $N\to\infty$ or $\Delta t \to 0$ and using the completeness relation (47) for the eigenvectors $|x\rangle$, one can approximately calculate the matrix element from (48),

$$\langle p^{(k)}| \exp\left\{-\frac{\mathrm{i}}{\hbar}\hat{H}\Delta t\right\}|p^{(k-1)}\rangle \approx \int \mathrm{d}x_{(k)} \langle p^{(k)}|1 - \frac{\mathrm{i}}{\hbar}\hat{H}\Delta t|x_{(k)}\rangle\langle x_{(k)}|p^{(k-1)}\rangle\,, \tag{49}$$

where $x_{(k)} = (x^i_{(k)})$ and $\mathrm{d}x_{(k)} = \prod_i \mathrm{d}x^i_{(k)}$. A result of this calculation can be expressed in terms of a classical Hamiltonian H; however, in the general case, it will depend on the choice of the correspondence rule between the classical function and quantum operator. For our calculations we choose the Weyl ordering. In this case the matrix element (49) will take the form

$$\int \frac{\mathrm{d}x_{(k)}}{(2\pi\hbar)^d} \exp\left\{\frac{\mathrm{i}}{\hbar}\left[-x^i_{(k)}\frac{p_i^{(k)} - p_i^{(k-1)}}{\Delta t} - H\left(x_{(k)} - \frac{\theta p^{(k)\prime}}{2\hbar}, p^{(k)\prime}\right)\right]\Delta t + O(\Delta t^2)\right\},$$

where $p^{(k)\prime} = \frac{p^{(k)} + p^{(k-1)}}{2}$, and $H\left(x - \frac{\theta p}{2\hbar}, p\right)$ is the Weyl symbol of the operator $\hat{H}$. Using the above formula and taking the limit $N\to\infty$ in the integral (48), we get for G_p the following path-integral representation:

$$G_p = \int_{p^{\rm in}}^{p^{\rm out}} \mathrm{D}p \int \mathrm{D}x \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left[-x_j\dot{p}^j - H\left(x - \frac{\theta p}{2\hbar}, p\right)\right]\right\}. \tag{50}$$

In the same manner, one can construct a path-integral representation for the evolution function G_x, which, is

$$G_x = \int \mathrm{D}p \int_{x_{\rm in}}^{x_{\rm out}} \mathrm{D}x \times \exp\left\{\frac{\mathrm{i}}{\hbar}\int \mathrm{d}t\left[p_j\dot{x}^j - H\left(x - \frac{\theta p}{2\hbar}, p\right)\right]\right\}. \tag{51}$$

Let us pass to the integration over trajectories $q = x - \frac{\theta p}{2\hbar}$ in the path integrals (50) and (51). Then we get

$$G_x = \int \mathrm{D}p \int_{x_{\rm in} - \theta p/2\hbar}^{x_{\rm out} - \theta p/2\hbar} \mathrm{D}q \exp\left\{\frac{\mathrm{i}}{\hbar} S^\theta_{\rm nonrel}\right\}, \tag{52}$$

$$G_p = \int_{p_{\rm in}}^{p_{\rm out}} \mathrm{D}p \int \mathrm{D}q \exp\left\{\frac{\mathrm{i}}{\hbar}\tilde{S}^\theta_{\rm nonrel}\right\}, \tag{53}$$

where

$$S^\theta_{\rm nonrel} = \int \mathrm{d}t\left[p_j\dot{q}^j - H(p,q) + \dot{p}_j\theta^{ji}p_i/2\hbar\right], \tag{54}$$

$$\tilde{S}^\theta_{\rm nonrel} = \int \mathrm{d}t\left[-q_j\dot{p}^j - H(p,q) - p_j\theta^{ji}\dot{p}_i/2\hbar\right]. \tag{55}$$

One ought to stress that the actions $S^\theta_{\rm nonrel}$ and $\tilde{S}^\theta_{\rm nonrel}$ differ by a total time derivative. The additional term in (54) reproduces, in 3D the one proposed in [26], and in 2D that put forward in [21].

The path integral (52) is a generalization of the result obtained in [14] for an arbitrary nonrelativistic system and without any restrictions on the matrix θ. One ought to say that path integrals on the noncommutative plane for matrix elements of the evolution operator in coherent state representations were studied in [16, 17]. They have specific forms that make it difficult to compare with our results.

In the conventional "commutative" nonsingular QM the action $S^\theta_{\rm nonrel}$ (at $\theta = 0$) is just the Hamiltonian action of the classical system under consideration. The canonical quantization of this action reproduces the initial QM of the system. In the noncommutative case this action is modified by a new term $\dot{p}_k\theta^{kj}p_j/2\hbar$. One can treat the action (54) as a θ-modified Hamiltonian action of the classical system under consideration (see the introduction). This interpretation can be justified by the canonical quantization of the action [21–23].

References

1. N. Seiberg, E. Witten, JHEP **9909**, 032 (1999)
2. H. Snyder, Phys. Rev. **71**, 38 (1947)
3. L. Alvarez-Gaume, S. Wadia, Phys. Lett. B **501**, 319 (2001)
4. M. Douglas, N. Nekrasov, Rev. Mod. Phys. **73**, 977 (2001)
5. A. Smailagic, E. Spalucci, Phys. Rev. D **65**, 107 701 (2002)
6. M. Demetrian, D. Kochan, hep-th/0102050
7. C. Duval, P. Horvathy, Phys. Lett. B **479**, 284 (2000)
8. V.P. Nair, A.P. Polychronakos, Phys. Lett. B **505**, 267 (2001)
9. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. **86**, 2716 (2001)
10. M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Lett. B **527**, 149 (2002)
11. M. Chaichian, A. Demichev, P. Presnajder, M.M. Sheikh-Jabbari, A. Tureanu, Nucl. Phys. B **611**, 383 (2001)
12. J. Gamboa, M. Loewe, J.C. Rojas, Phys. Rev. D **64**, 067 901 (2001)
13. G. Mangano, J. Math. Phys. **39**, 2584 (1998)
14. C. Acatrinei, JHEP **0109**, 007 (2001)

15. B. Dragovich, Z. Rakic, Theor. Math. Phys. **140**, 1299 (2004)
16. A. Smailagic, E. Spalucci, J. Phys. A **36**, 467 (2003)
17. H. Tan, J. Phys. A **39**, 15 299 (2006)
18. E.S. Fradkin, D.M. Gitman, Phys. Rev. D **44**, 3230 (1991)
19. D.M. Gitman, Nucl. Phys. B **468**, 490 (1997)
20. F.A. Berezin, M.S. Marinov, Ann. Phys. **104**, 336 (1977)
21. C. Duval, P. Horvathy, J. Phys. A **34**, 10 097 (2001)
22. A.A. Deriglazov, Phys. Lett. B **555**, 83 (2003)
23. A.A. Deriglazov, JHEP **0303**, 021 (2003)
24. M. Chaichian, P.P. Kulish, K. Nishijima, A. Tureanu, Phys. Lett. B **604**, 98 (2004)
25. M. Chaichian, P. Presnajder, A. Tureanu, Phys. Rev. Lett. **94**, 151 602 (2005)
26. M.-C. Chang, Q. Niu, Phys. Rev. Lett. **75**, 1348 (1995)
27. J. Schwinger, Phys. Rev. **82**, 664 (1951)
28. R. Brauer, H. Weyl, Am. J. Math. **57**, 425 (1935)
29. S. Bourouaine, A. Benslama, J. Phys. A **38**, 7389 (2005)
30. F.A. Berezin, The Method of Second Quantization (Nauka, Moscow, 1965)
31. F.A. Berezin, Introduction to Algebra and Analysis with Anticommuting Variables (Moscow State University Press, Moscow, 1983)
32. F.A. Berezin, Introduction to Superanalysis (D. Reidel, Dordrecht, 1987)
33. D.M. Gitman, I.V. Tyutin, Quantization of Fields with Constraints (Springer, Amsterdam, 1990)
34. P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University Press, New York, 1964)
35. A. Pinzul, A. Stern, Phys. Lett. B **593**, 279 (2004)